ON THE MULTIMOMENTUM BUNDLES AND THE LEGENDRE MAPS IN FIELD THEORIES

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Abstract

We study the geometrical background of the Hamiltonian formalism of first-order Classical Field Theories. In particular, different proposals of *multimomentum bundles* existing in the usual literature (including their canonical structures) are analyzed and compared. The corresponding *Legendre maps* are introduced. As a consequence, the definition of *regular* and *almost-regular* Lagrangian systems is reviewed and extended from different but equivalent ways.

Key words: Jet Bundles, Classical Field Theories, Legendre map, Hamiltonian formalism.

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1 Introduction

The standard geometric structures underlying the covariant Lagrangian description of first-order Field Theories are first order jet bundles $J^1E \xrightarrow{\pi^1} E \xrightarrow{\pi} M$ and their canonical structures [5], [7], [9], [18], [19], [23], [28]. Nevertheless, for the covariant Hamiltonian formalism of these theories there are several choices for the phase space where this formalism takes place. Among all of them, only the *multisymplectic* models will deserve our attention in this work. So, in [11], [12], [17] and [24], (see also [15], [16] and [17]) the *multimomentum phase space* is taken to be $\mathcal{M}\pi \equiv \Lambda_1^m\mathrm{T}^*E$, the bundle of m-forms on E (m being the dimension of M) vanishing by the action of two π-vertical vector fields. In [3], [14], [20] and [21] use is made of $J^1\pi^* \equiv \Lambda_1^m\mathrm{T}^*E/\Lambda_0^m\mathrm{T}^*E$ as the multimomentum phase space (where $\Lambda_0^m\mathrm{T}^*E$ is the bundle of π-semibasic m-forms in E). Finally, in [4], [8], [9], [25], and [26] the basic choice is the bundle $\Pi \equiv \pi^*\mathrm{T}M \otimes \mathrm{V}^*(\pi) \otimes \pi^*\Lambda^m\mathrm{T}^*M$ (here $\mathrm{V}^*(\pi)$ denotes the dual bundle of $\mathrm{V}(\pi)$: the π-vertical subbundle of $\mathrm{T}E$) which, in turns, is related to $J^1E^* \equiv \pi^*\mathrm{T}M \otimes \mathrm{T}^*E \otimes \pi^*\Lambda^m\mathrm{T}^*M$. The origin of all these multimomentum bundles is related to the different *Legendre maps* which arise essentially from the fiber derivative of the Lagrangian density (see Section 4).

All these choices have special features. So, $\mathcal{M}\pi$ and J^1E^* are endowed with natural multisymplectic forms. In works such as [12], the former is used for obtaining the *Poincaré-Cartan* form in J^1E, which is needed for the Lagrangian formalism. This is done by defining the suitable *Legendre map* connecting J^1E and $\mathcal{M}\pi$. On the other hand, the dimensions of J^1E and $\mathcal{M}\pi$ and J^1E^* are not equal: in fact, $\dim \mathcal{M}\pi = \dim J^1E + 1$ and $\dim J^1E^* = \dim J^1E + (\dim M)^2$; but $\dim \Pi = \dim J^1\pi^* = \dim J^1E$ and the choice of both $J^1\pi^*$ and Π as multimomentum phase spaces allows us to state coherent covariant Hamiltonian formalisms for Field Theories. Finally, the construction of $J^1\pi^*$ is closely related to $\mathcal{M}\pi$, but their relation to Π and J^1E^* is not evident at all.

Hence, the aim of this work is to carry out a comparative study of these multimomentum bundles, and introduce the canonical geometrical structures of $\mathcal{M}\pi$ and J^1E^*. In every case, the corresponding *Legendre map* is also defined. An interesting conclusion of this study is that the multimomentum bundles $J^1\pi^*$ and Π are canonically diffeomorphic. Finally, using the different Legendre maps, we can classify Lagrangian systems in Field Theory into *(hyper)regular* and *almost-regular*, from different but equivalent ways, attending to the characteristics of these maps.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood.

2 Multimomentum bundles

Let $\pi\colon E \to M$ be a fiber bundle ($\dim E = N + m$, $\dim M = m$), $\pi^1\colon J^1E \to E$ the first order jet bundle of local sections of π, and $\bar{\pi}^1 = \pi \circ \pi^1$. J^1E is an affine bundle modeled on $\pi^*\mathrm{T}^*M \otimes \mathrm{V}(\pi)$.

A local chart of natural coordinates in E adapted to the bundle $E \to M$ will be denoted by (x^μ, y^A). The induced local chart in J^1E is denoted by (x^μ, y^A, v_μ^A).

Definition 1 *The bundle (over E)*

$$J^1E^* := \pi^*\mathrm{T}M \otimes_E \mathrm{T}^*E \otimes_E \pi^*\Lambda^m\mathrm{T}^*M$$

is called the generalized multimomentum bundle *associated with the bundle $\pi\colon E \to M$. We denote the natural projections by $\hat{\rho}^1\colon J^1E^* \to E$ and $\bar{\hat{\rho}}^1 := \pi \circ \hat{\rho}^1\colon J^1E^* \to M$.*

The local system (x^μ, y^A) induces a local system of natural coordinates $(x^\mu, y^A, \mathrm{p}^\nu_\mu, \mathrm{p}^\mu_A)$ in J^1E^* as follows: if $\mathbf{y} \in J^1E^*$, with $\mathbf{y} \stackrel{\hat{\rho}^1}{\rightarrow} y \stackrel{\pi}{\rightarrow} x$, we have that

$$\mathbf{y} = \frac{\partial}{\partial x^\mu}\Big|_y \otimes (f^\mu_\nu \mathrm{d}x^\nu + g^\mu_A \mathrm{d}y^A)_y \otimes \mathrm{d}^m x|_y \tag{1}$$

(where $\mathrm{d}^m x \equiv \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$), and therefore

$$\begin{array}{ccc} x^\mu(\mathbf{y}) = x^\mu((\pi \circ \hat{\rho}^1)(\mathbf{y})) & ; & y^A(\mathbf{y}) = y^A(\hat{\rho}^1(\mathbf{y})) \\ \mathrm{p}^\mu_\nu(\mathbf{y}) = \mathbf{y}\left(\mathrm{d}x^\mu \otimes \frac{\partial}{\partial x^\nu} \otimes \partial^m x\Big|_y\right) = f^\mu_\nu & ; & \mathrm{p}^\mu_A(\mathbf{y}) = \mathbf{y}\left(\mathrm{d}x^\mu \otimes \frac{\partial}{\partial y^A} \otimes \partial^m x\Big|_y\right) = g^\mu_A \end{array}$$

(where $\partial^m x \equiv \dfrac{\partial}{\partial x^1} \wedge \ldots \wedge \dfrac{\partial}{\partial x^m}$).

Definition 2 *The bundle (over E)*

$$\Pi := \pi^*\mathrm{T}M \otimes_E \mathrm{V}^*(\pi) \otimes_E \pi^*\Lambda^m\mathrm{T}^*M = \bigcup_{y\in E} \mathrm{T}_{\pi(y)}M \otimes \mathrm{V}^*_y(\pi) \otimes \Lambda^m\mathrm{T}^*_{\pi(y)}M$$

is called the reduced multimomentum bundle *associated with the bundle $\pi\colon E \to M$. We denote the natural projections by $\rho^1\colon \Pi \to E$ and $\bar{\rho}^1 := \pi \circ \rho^1\colon \Pi \to M$.*

From the local system (x^μ, y^A) we can construct a natural system of coordinates $(x^\mu, y^A, \mathrm{p}^\mu_A)$ in Π as follows: considering $\tilde{y} \in \Pi$, we write

$$x^\mu(\tilde{y}) = x^\mu(\bar{\rho}^1(\tilde{y}))\ ;\ y^A(\tilde{y}) = y^A(\rho^1(\tilde{y}))\ ;\ \mathrm{p}^\mu_A(\tilde{y}) = \tilde{y}\left(\mathrm{d}x^\mu, \frac{\partial}{\partial y^A}, \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^m}\right)$$

and denoting the dual basis of $\dfrac{\partial}{\partial y^A}$ in $\mathrm{V}^*(\pi)$ by $\{\zeta^A\}$, an element $\tilde{y} \in \Pi$ is expressed as

$$\tilde{y} = \mathrm{p}^\mu_A(\tilde{y}) \frac{\partial}{\partial x^\mu} \otimes \zeta^A \otimes \mathrm{d}^m x\Big|_{(x^\mu(\tilde{y}), y^A(\tilde{y}))}$$

The relation between these multimomentum bundles is given by the (onto) map

$$\begin{array}{cccccc} \delta & : & J^1E^* & \longrightarrow & \Pi \\ & & (x^\mu, y^A, \mathrm{p}^\mu_A, \mathrm{p}^\nu_\mu) & \mapsto & (x^\mu, y^A, \mathrm{p}^\mu_A) \end{array}$$

which is induced by the natural restriction $\mathrm{T}^*E \to \mathrm{V}^*(\pi)$.

Definition 3 *Consider the multicotangent bundle $\Lambda^m\mathrm{T}^*E$. Then, for every $y \in E$ we define*

$$\Lambda^m_1\mathrm{T}^*_yE := \{\gamma \in \Lambda^m\mathrm{T}^*_yE\ ;\ i(u_1)\, i(u_2)\gamma = 0\ ,\ u_1, u_2 \in \mathrm{V}_y(\pi)\}$$

The bundle (over E)

$$\mathcal{M}\pi \equiv \Lambda^m_1\mathrm{T}^*E := \bigcup_{y\in E} \Lambda^m_1\mathrm{T}^*_yE = \bigcup_{y\in E} \{(y, \alpha)\ ;\ \alpha \in \Lambda^m_1\mathrm{T}^*_yE\}$$

will be called the extended multimomentum bundle *associated with the bundle $\pi\colon E \to M$. We denote the natural projections by $\hat{\kappa}^1\colon \mathcal{M}\pi \to E$ and $\bar{\hat{\kappa}}^1\colon \mathcal{M}\pi \to M$.*

The local chart (x^μ, y^A) in E induces a natural system of coordinates (x^μ, y^A, p, p^μ_A) in $\mathcal{M}\pi$. Hence, if $\hat{y} \in \mathcal{M}\pi$ (with $\hat{y} \xrightarrow{\hat{\kappa}^1} y \xrightarrow{\pi} x$), it is a m-covector whose expressions in a natural chart is

$$\hat{y} = \lambda\, \mathrm{d}^m x + \lambda^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu$$

and we have that

$$x^\mu(\hat{y}) = x^\mu(y)\ ,\ y^A(\hat{y}) = y^A(y)\ ,\ p(\hat{y}) = \hat{y}(\partial^m x) = \lambda\ ,\ p^\mu_A(\hat{y}) = \hat{y}\left(\frac{\partial}{\partial y^A} \wedge \partial^{m-1}x^\mu\right) = \lambda^\mu_A$$

(where $\partial^{m-1}x^\mu \equiv i(\mathrm{d}x^\mu)\partial^m x$).

Another usual characterization of the bundle $\mathcal{M}\pi$ is the following:

Proposition 1 *$\mathcal{M}\pi \equiv \Lambda^m_1 \mathrm{T}^*E$ is canonically isomorphic to* $\mathrm{Aff}(J^1E, \pi^*\Lambda^m\mathrm{T}^*M)$*.*

(*Proof*) It is a consequence of Lemma 3 of the appendix, taking $G = \mathrm{T}_yE$, $H = \mathrm{T}_xM$, $F = \mathrm{V}_y(\pi)$, and $\Sigma = J^1_yE$, and then observing that the sequence (6) is

$$0 \longrightarrow \mathrm{V}_y(\pi) \xrightarrow{j_y} \mathrm{T}_yE \xrightarrow{\mathrm{T}_y\pi} \mathrm{T}_xM \longrightarrow 0$$

(See also [3]). ∎

Comment:

- Given a section $\mathit{\Gamma}\colon E \to J^1E$ of π^1, in the same way as it is commented in the Lemma 2 of the appendix, we have an splitting

  $$\mathrm{Aff}\,(J^1E, \pi^*\Lambda^m\mathrm{T}^*M) \simeq \pi^*\Lambda^m\mathrm{T}^*M \oplus (\pi^*\Lambda^m\mathrm{T}^*M \otimes \mathrm{V}^*(\pi))$$

  and them $\dim\,(\mathcal{M}\pi)_y = \dim\,\Pi_y + 1$, for every $y \in E$.

We can introduce the *canonical contraction* in J^1E^*, which is defined as the map

$$\begin{array}{ccccc} \iota & : & J^1E^* \equiv \pi^*\mathrm{T}M \otimes \mathrm{T}^*E \otimes \pi^*\Lambda^m\mathrm{T}^*M & \longrightarrow & \Lambda^m\mathrm{T}^*E \\ & & \mathbf{y} = u_k \otimes \alpha^k \otimes \chi & \mapsto & \alpha^k \wedge \pi^*\, i(u_k)\chi \end{array}$$

In a natural chart $(x^\mu, y^A, \mathrm{p}^\nu_\mu, \mathrm{p}^\mu_A)$ in J^1E^*, bearing in mind (1), we obtain

$$\iota(\mathbf{y}) = (f^\mu_\nu \mathrm{d}x^\nu + g^\mu_A \mathrm{d}y^A)_y \wedge i\left(\frac{\partial}{\partial x^\mu}\right) \mathrm{d}^m x\Big|_y = (f^\mu_\mu \mathrm{d}^m x + g^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu)_y$$

(where $\mathrm{d}^{m-1}x_\mu \equiv i\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^m x$. Remember that f^μ_μ denotes $\displaystyle\sum_{\mu=1}^m f^\mu_\mu$).

Therefore, the relation between the multimomentum bundles $\mathcal{M}\pi$ and J^1E^* is:

Proposition 2 $\mathcal{M}\pi = \iota(J^1E^*)$*. (We will denote $\iota_0\colon J^1E^* \to \mathcal{M}\pi$ the restriction of ι onto its image $\mathcal{M}\pi$).*

(*Proof*) For every $y \in E$ we must prove that

$$\iota(J^1E^*)_y = \{\gamma \in \Lambda^m\mathrm{T}^*_yE\ ;\ i(u_1)\, i(u_2)\gamma = 0\ ,\ u_1, u_2 \in \mathrm{V}_y(\pi)\} \equiv \Lambda^m_1\mathrm{T}^*_yE$$

In fact; if $\mathbf{y} \in J^1E^*$, $\gamma = \iota(\mathbf{y})$ and $u_1, u_2 \in \mathrm{V}_y\pi$, in a local chart we have

$$i(u_1)\, i(u_2)(\iota(\mathbf{y})) = i(u_1)\, i(u_2)(\mathrm{p}_\nu^\nu(\mathbf{y})\mathrm{d}^m x + p_A^\mu(\mathbf{y})\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu)_y = 0$$

and conversely, if $\gamma \in \Lambda^m\mathrm{T}_y^*E$ satisfies the above condition, then $\gamma = (\lambda \mathrm{d}^m x + \lambda_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu)_y$, therefore $\gamma = \iota(\mathbf{y})$, with

$$\mathbf{y} = \frac{\partial}{\partial x^\mu}\Big|_y \otimes \left(\left(\frac{\lambda}{m}(\mathrm{d}x^1 + \ldots + \mathrm{d}x^m) + \lambda_A^\mu \mathrm{d}y^A\right)\Big|_y \otimes \mathrm{d}^m x|_y\right)$$

(Observe that $\Lambda_1^m\mathrm{T}^*E$ is canonically isomorphic to $\mathrm{T}^*E \wedge \pi^*\Lambda^{m-1}\mathrm{T}^*M$). ■

Note that, in natural coordinates, we have

$$\iota\colon \mathbf{y} \equiv (x^\mu, y^A, \mathrm{p}_A^\mu, \mathrm{p}_\mu^\nu) \mapsto \hat{y} \equiv (x^\mu, y^A, \mathrm{p}_A^\mu, \mathrm{p} = p_\mu^\mu)$$

The sections of the bundle $\pi^*\Lambda^m\mathrm{T}^*M \to E$ are the π-semibasic m-forms on E. Therefore we introduce the notation $\Lambda_0^m\mathrm{T}^*E \equiv \pi^*\Lambda^m\mathrm{T}^*M$, and then:

Definition 4 *The bundle (over E)*

$$J^1\pi^* := \Lambda_1^m\mathrm{T}^*E/\Lambda_0^m\mathrm{T}^*E \equiv \mathcal{M}\pi/\Lambda_0^m\mathrm{T}^*E$$

will be called the restricted multimomentum bundle *associated with the bundle $\pi\colon E \to M$. We denote the natural projections by $\kappa^1\colon J^1\pi^* \to E$ and $\bar{\kappa}^1 := \pi \circ \kappa^1\colon J^1\pi^* \to M$.*

The natural coordinates in $J^1\pi^*$ will be denoted as (x^μ, y^A, p_A^μ).

The relation between the bundles $\mathcal{M}\pi$ and $J^1\pi^*$ is given by the natural projection

$$\begin{array}{cccc} \mu & : & \mathcal{M}\pi = \Lambda_1^m\mathrm{T}^*E & \longrightarrow & \Lambda_1^m\mathrm{T}^*E/\Lambda_0^m\mathrm{T}^*E = J^1\pi^* \\ & & (x^\mu, y^A, p_A^\mu, p) & \mapsto & (x^\mu, y^A, p_A^\mu) \end{array}$$

Finally, the relation between the multimomentum bundles Π and $J^1\pi^*$ is:

Theorem 1 *The multimomentum bundles $J^1\pi^*$ and Π are canonically diffeomorphic. We will denote this diffeomorphism by $\Psi\colon J^1\pi^* \to \Pi$.*

(*Proof*) By Proposition 1, we have that $\Lambda_1^m\mathrm{T}^*E \equiv \mathcal{M}\pi$ is canonically isomorphic to $\mathrm{Aff}(J^1E, \Lambda^m\mathrm{T}^*M)$. On the other hand, taking $G = \mathrm{T}_yE$, $H = \mathrm{T}_xM$, $F = \mathrm{V}_y(\pi)$, and $\Sigma = J_y^1E$ in Lemma 4 of the appendix, we obtain

$$\mathrm{Aff}(J_y^1E, \Lambda^m\mathrm{T}_x^*M)/\Lambda^m\mathrm{T}_x^*M \ \simeq\ \mathrm{T}_xM \otimes \mathrm{V}_y^*(\pi) \otimes \Lambda^m\mathrm{T}_x^*M$$

Therefore, extending these constructions to the bundles we have

$$J^1\pi^* \ \simeq\ \mathrm{Aff}(J^1E, \pi^*\Lambda^m\mathrm{T}^*M)/\pi^*\Lambda^m\mathrm{T}^*M \ \simeq\ \pi^*\mathrm{T}M \otimes \mathrm{V}^*(\pi) \otimes \pi^*\Lambda^m\mathrm{T}^*M := \Pi$$

■

Remark:

- As is known, a connection in the bundle $\pi: E \to M$, that is, a section $\mathit{\Gamma}: E \to J^1E$ of π^1, induces a linear map $\bar{\mathit{\Gamma}}: \mathrm{V}^*(\pi) \to \mathrm{T}^*E$ and, as a consequence, another one

$$\begin{array}{rcccc} \tilde{\mathit{\Gamma}} & : & \pi^*\mathrm{T}M \otimes \mathrm{V}^*(\pi) \otimes \Lambda^m\mathrm{T}^*M & \to & \Lambda_1^m\mathrm{T}^*E \\ & & u \otimes \alpha \otimes \xi & \mapsto & \bar{\mathit{\Gamma}}(\alpha) \wedge i(u)\xi \end{array}$$

  In this way, we can get the inverse map Ψ^{-1} by means of a connection: it is the composition of $\tilde{\mathit{\Gamma}}$ with $\mu: \Lambda_1^m\mathrm{T}^*E \to \Lambda_1^m\mathrm{T}^*E/\Lambda_0^m\mathrm{T}^*E$. Nevertheless, Ψ^{-1} is connection independent because, given two connections $\mathit{\Gamma}_1, \mathit{\Gamma}_2$, the image of $\mathit{\Gamma}_1 - \mathit{\Gamma}_2$ is in $\Lambda_0^m\mathrm{T}^*E \subset \Lambda_1^m\mathrm{T}^*E$.

Next, we give the coordinate expression of this diffeomorphism. First, let us recall that the natural coordinates in E produce an affine reference frame in J^1E as follows: if $\pi(y) = x$, take $\bar{y}_0, \bar{y}_A^\mu \in J_y^1E$ as

$$\begin{aligned} \bar{y}_0 &= \{\phi: M \to E \;;\; \phi(x) = y \;,\; \mathrm{T}_x\phi\left(\frac{\partial}{\partial x^\mu}\right)_x = \left(\frac{\partial}{\partial x^\mu}\right)_y\} \\ \bar{y}_A^\mu &= \{\phi: M \to E \;;\; \phi(x) = y \;,\; \mathrm{T}_x\phi\left(\frac{\partial}{\partial x^\mu}\right)_x = \frac{\partial}{\partial x^\mu}\Big|_y + \frac{\partial}{\partial y^A}\Big|_y\} \end{aligned}$$

(Observe that $\bar{y}_0$ is the 1-jet of critical sections with target y); then we obtain $\bar{y} - \bar{y}_0 = v_\mu^A(\bar{y})(\bar{y}_A^\mu - \bar{y}_0)$, for every $\bar{y} \in J_y^1E$.

An affine map $\varphi: J^1E \to \Lambda_0^m\mathrm{T}^*E$ is given by $\varphi = (\varphi(\bar{y}_0), \hat{\varphi})$ (where $\hat{\varphi}$ denotes the linear part of φ), then

$$\varphi(\bar{y}) = \varphi(\bar{y}_0) + \hat{\varphi}(v_\mu^A(\bar{y})(\bar{y}_A^\mu - \bar{y}_0)) = \varphi(\bar{y}_0) + v_\mu^A(\bar{y})\hat{\varphi}(\bar{y}_A^\mu - \bar{y}_0)$$

Denoting by q the fiber coordinate in $\pi^*\Lambda^m\mathrm{T}^*M \equiv \Lambda_0^m\mathrm{T}^*E$. We have

$$q(\varphi(\bar{y})) = (\varphi(\bar{y}_0))(\partial^m x) + v_\mu^A(\bar{y})\hat{\varphi}(\bar{y}_A^\mu - \bar{y}_0)(\partial^m x) = \lambda + v_\mu^A(\bar{y})\lambda_A^\mu$$

Then we have an affine coordinate system in $\mathrm{Aff}(J^1E, \Lambda_0^m\mathrm{T}^*E)$, denoted (q, q_A^μ), with

$$q(\varphi) = q(\varphi(\bar{y}_0)) \quad , \quad q_A^\mu(\varphi) = q(\hat{\varphi}(\bar{y}_A^\mu - \bar{y}_0))$$

and hence $\varphi = (l, l_A^\mu)$, with $l = q(\varphi)$, $l_A^\mu = q_A^\mu(\varphi)$ or, what means the same thing, $\varphi(v_\mu^A) = (l + l_A^\mu v_\mu^A)\mathrm{d}^m x$.

Now, let $\bar{y} \in J^1E$, with $\bar{y} \xrightarrow{\pi^1} y \xrightarrow{\pi} x$. Consider the map

$$\begin{array}{rcccccl} \Upsilon & : & \Lambda_1^m\mathrm{T}^*E & \longrightarrow & \mathrm{Aff}(J^1E, \Lambda_0^m\mathrm{T}^*E) & & \\ & & \eta & \mapsto & \Upsilon(\eta) & : & \bar{y} \mapsto (\phi^*\eta)_y \end{array}$$

(introduced in Lemma 3 of the appendix), where $\phi: M \to E$ is a representative of $\bar{y}$, with $\phi(x) = y$. If $\beta = (\lambda\mathrm{d}^m x + \lambda_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu)_y \in \Lambda_1^m\mathrm{T}^*E$, then

$$\mathrm{T}_x\phi = \begin{pmatrix} \mathrm{Id} \\ v_\mu^A(y) \end{pmatrix} \quad , \quad \mathrm{T}_x\phi\left(\frac{\partial}{\partial x^\mu}\right)_x = \left(\frac{\partial}{\partial x^\mu}\right)_y + v_\mu(\bar{y})\left(\frac{\partial}{\partial y^A}\right)_y$$

Therefore we obtain

$$(\Upsilon(\beta))(\bar{y}) = (\phi^*\beta)_y = ((\lambda + \lambda_A^\mu v_\mu^A(\bar{y}))\mathrm{d}^m x)_y$$

that is, $\Upsilon(\beta) = (\lambda, \lambda_A^\mu)$ or, what is equivalent,

$$q(\Upsilon(\beta)) = p(\beta) = \lambda \;,\; q_A^\mu(\Upsilon(\beta)) = p_A^\mu(\beta) = \lambda_A^\mu \quad \text{or} \quad \Upsilon^*q = p \;,\; \Upsilon^*q_A^\mu = p_A^\mu$$

Thus Υ is a diffeomorphism in the fiber, and therefore a diffeomorphism, since it is the identity on the base.

Finally, we have the (commutative) diagram

$$
\begin{array}{ccc}
\Lambda^m_1 \mathrm{T}^*E & \xrightarrow{\Upsilon} & \mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E) \\
\downarrow & & \downarrow \\
\Lambda^m_1 \mathrm{T}^*E/\Lambda^m_0 \mathrm{T}^*E & \xrightarrow{\Psi} & \mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E)/\Lambda^m_0 \mathrm{T}^*E
\end{array}
$$

and, for proving that Υ goes to the quotient, it suffices to see that $\Upsilon(\Lambda^m_0 \mathrm{T}^*E) \subset \Lambda^m_0 \mathrm{T}^*E$. To show this we must identify $\Lambda^m_0 \mathrm{T}^*E$ as a subset of both $\Lambda^m_1 \mathrm{T}^*E$ and $\mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E)$. In $\Lambda^m_1 \mathrm{T}^*E$ we have that the elements of $\Lambda^m_0 \mathrm{T}^*E$ are characterized as follows: $\beta \in \Lambda^m_0 \mathrm{T}^*E$ iff, for a natural coordinate system (x^μ, y^A, p, p^μ_A), we have $p^\mu_A(\beta) = 0$. On the other hand, as a subset of $\mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E)$,

$$
\Lambda^m_0 \mathrm{T}^*E \equiv \{\varphi \in \mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E) \ ; \ \varphi = \text{constant}\} = \{\varphi \in \mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E) \ ; \ \hat{\varphi} = 0\}
$$

($\hat{\varphi}$ denotes the linear part of φ), or equivalently, for an affine natural coordinate system (q, q^μ_A),

$$
\Lambda^m_0 \mathrm{T}^*E \equiv \{\varphi \in \mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E) \ ; \ q^\mu_A(\varphi) = 0\}
$$

And, from the local expression of Υ, we obtain that $\Upsilon(\Lambda^m_0 \mathrm{T}^*E) = \Lambda^m_0 \mathrm{T}^*E$. As a consequence of this, if p^μ_A are the fiber coordinates in $J^1\pi^* = \Lambda^m_1 \mathrm{T}^*E/\Lambda^m_0 \mathrm{T}^*E$, and p^μ_A are those in $\Pi = \mathrm{Aff}(J^1E, \Lambda^m_0 \mathrm{T}^*E)/\Lambda^m_0 \mathrm{T}^*E$, we have that

$$
\Psi^* \mathrm{p}^\mu_A = p^\mu_A \quad , \quad \forall \mu, A
$$

and, consequently, in these natural coordinate systems, the diffeomorphism Ψ is the identity.

3 Canonical forms in the multimomentum bundles

The multimomentum bundles J^1E^* and $\mathcal{M}\pi$ are endowed with canonical differential forms:

Definition 5 *The* canonical m-form *of J^1E^* is the form $\hat{\Theta} \in \mathit{\Omega}^m(J^1E^*)$ defined as follows: if $\mathbf{y} \in J^1E^*$ and $X_1, \ldots, X_m \in \mathfrak{X}(J^1E^*)$, then*

$$
\hat{\Theta}(\mathbf{y}; X_1, \ldots, X_m) := \iota(\mathbf{y})(\mathrm{T}_{\mathbf{y}}\hat{\rho}^1(X_1), \ldots, \mathrm{T}_{\mathbf{y}}\hat{\rho}^1(X_m))
$$

The canonical $(m+1)$-form *of J^1E^* is $\hat{\Omega} := -\mathrm{d}\hat{\Theta} \in \mathit{\Omega}^{m+1}(J^1E^*)$.*

In a natural chart in J^1E^* we have

$$
\begin{aligned}
\hat{\Theta} &= \mathrm{p}^\nu_\nu \mathrm{d}^m x + \mathrm{p}^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu \\
\hat{\Omega} &= -\mathrm{dp}^\nu_\nu \wedge \mathrm{d}^m x - \mathrm{dp}^\mu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu
\end{aligned}
$$

Remark:

- As is known [2], the *multicotangent bundle* $\Lambda^m \mathrm{T}^*E$ is endowed with canonical forms: $\mathbf{\Theta} \in \mathit{\Omega}^m(\Lambda^m \mathrm{T}^*E)$ and the multisymplectic form $\mathbf{\Omega} := -\mathrm{d}\mathbf{\Theta} \in \mathit{\Omega}^{m+1}(\Lambda^m \mathrm{T}^*E)$. Then, it can be easily proved that $\hat{\Theta} = \iota^*\mathbf{\Theta}$.

Observe that $\mathcal{M}\pi \equiv \Lambda^m_1 \mathrm{T}^*E$ is a subbundle of the *multicotangent bundle* $\Lambda^m \mathrm{T}^*E$. Let $\varsigma \colon \Lambda^m_1 \mathrm{T}^*E \hookrightarrow \Lambda^m \mathrm{T}^*E$ be the natural imbedding (hence $\varsigma \circ \iota_0 = \iota$). Then:

Definition 6 *The* canonical m-form *of* $\mathcal{M}\pi$ *is* $\Theta := \varsigma^*\boldsymbol{\Theta} \in \Omega^m(\mathcal{M}\pi)$. *It is also called the* multimomentum Liouville m-form.

The canonical $(m+1)$-form *of* $\mathcal{M}\pi$ *is* $\Omega = -\mathrm{d}\Theta \in \Omega^{m+1}(\mathcal{M}\pi)$, *and it is called the* multimomentum Liouville $(m+1)$-form.

The expressions of these forms in a natural chart in $\mathcal{M}\pi$ are

$$\begin{aligned}\Theta &= p\mathrm{d}^m x + p_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu \\ \Omega &= -\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p_A^\mu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu\end{aligned}$$

Then, a simple calculation allows us to prove that Ω is 1-nondegenerate, and hence $(\mathcal{M}\pi, \Omega)$ is a *multisymplectic manifold*.

Remark:

- Considering the natural projection $\hat{\kappa}^1 \colon \Lambda_1^m \mathrm{T}^*E \to E$, then

$$\Theta((y,\alpha); X_1, \ldots, X_m) := \alpha(y; \mathrm{T}_{(y,\alpha)}\hat{\kappa}^1(X_1), \ldots, \mathrm{T}_{(y,\alpha)}\hat{\kappa}^1(X_m))$$

for every $(y,\alpha) \in \Lambda_1^m \mathrm{T}^*E$ (where $y \in E$ and $\alpha \in \Lambda_1^m \mathrm{T}_y^*E$), and $X_i \in \mathfrak{X}(\Lambda_1^m \mathrm{T}^*E)$.

The canonical m-forms $\hat{\Theta}$ and Θ can be characterized alternatively in the following way:

Proposition 3 *1.* $\hat{\Theta}$ *is the unique* $\hat{\rho}^1$*-semibasic* m*-form in* J^1E^* *such that, for every section* $\hat{\psi} \colon E \to J^1E^*$ *of* $\hat{\rho}$*, the relation* $\hat{\psi}^*\hat{\Theta} = \iota \circ \hat{\psi}$ *holds.*

2. Θ *is the only* $\hat{\kappa}^1$*-semibasic* m*-form in* $\mathcal{M}\pi$ *such that, for every section* $\phi \colon E \to \Lambda_1^m \mathrm{T}^*E$ *of* $\hat{\kappa}^1$*, the relation* $\phi^*\Theta = \phi$ *holds.*

(*Proof*)

1. From the definition of $\hat{\Theta}$ it is obvious that it is $\hat{\rho}^1$-semibasic. Then, for the second relation, let $y \in E$ and $u_1, \ldots, u_m \in \mathrm{T}_yE$; therefore

$$\begin{aligned}(\hat{\psi}^*\hat{\Theta})(y; u_1, \ldots, u_m) &= \hat{\Theta}(\hat{\psi}(y); \mathrm{T}_y\hat{\psi}(u_1), \ldots, \mathrm{T}_y\hat{\psi}(u_m)) \\ &= \iota(\hat{\psi}(y))(y; (\mathrm{T}_{\hat{\psi}(y)}\hat{\rho}^1 \circ \mathrm{T}_y\hat{\psi})(u_1), \ldots, (\mathrm{T}_{\hat{\psi}(y)}\hat{\rho}^1 \circ \mathrm{T}_y\hat{\psi})(u_m)) \\ &= \iota(\hat{\psi}(y))(y; u_1, \ldots, u_m) = (\iota \circ \hat{\psi})(y; u_1, \ldots, u_m)\end{aligned}$$

Conversely, suppose that $\hat{\Theta} \in \Omega^m(J^1E^*)$ verifies both conditions in the statement. We will prove that $\hat{\Theta}$ is uniquely determined. Let $\mathbf{y} \in J^1E^*$, with $\mathbf{y} \xrightarrow{\hat{\rho}^1} y \xrightarrow{\pi} x$, and $v_1, \ldots, v_m \in \mathrm{T}_{\mathbf{y}}J^1E^*$. We can suppose that $v_1, \ldots, v_m$ are linearly independent and that
$\langle v_1, \ldots, v_m \rangle \cap \mathrm{V}_{\bar{y}}(\pi \circ \hat{\rho}^1) = \{0\}$ (where $\langle v_1, \ldots, v_m \rangle$ denotes the subspace generated by these vectors), for in other case $\hat{\Theta}(\mathbf{y}; v_1, \ldots, v_m) = 0$. Then, by the Lemma 1 (see the appendix), there exist $u_1, \ldots, u_m \in \mathrm{T}_xM$ and a local section $\hat{\psi} \colon M \to J^1E^*$ of $\pi \circ \hat{\rho}^1$, such that $\hat{\psi}(x) = \mathbf{y}$ and $\mathrm{T}_x\hat{\psi}(u_\mu) = v_\mu$ $(\mu = 1, \ldots, m)$, and we have that

$$\begin{aligned}\hat{\Theta}(\mathbf{y}; v_1, \ldots, v_m) &= \hat{\Theta}(\hat{\psi}(y); \mathrm{T}_y\hat{\psi}(u_1), \ldots, \mathrm{T}_y\hat{\psi}(u_m)) \\ &= (\hat{\psi}^*\hat{\Theta})(y; u_1, \ldots, u_m) = (\iota \circ \hat{\psi})(y; u_1, \ldots, u_m)\end{aligned}$$

Hence $\hat{\Theta}(\mathbf{y}; v_1, \ldots, v_m)$ is uniquely determined.

2. The proof follows the same pattern as the one above.

■

4 Legendre maps

From the Lagrangian point of view, a *classical Field Theory* is described by its *configuration bundle* $\pi\colon E \to M$ (where M is an oriented manifold with volume form $\omega \in \Omega^m(M)$); and a *Lagrangian density* which is a $\bar{\pi}^1$-semibasic m-form on J^1E. A Lagrangian density is usually written as $\mathcal{L} = \pounds(\bar{\pi}^{1^*}\omega)$, where $\pounds \in \mathrm{C}^\infty(J^1E)$ is the *Lagrangian function* associated with $\mathcal{L}$ and ω. Then a *Lagrangian system* is a couple $((E,M;\pi),\mathcal{L})$. The *Poincaré-Cartan* m *and* $(m+1)$*-forms* associated with the Lagrangian density $\mathcal{L}$ are defined using the *vertical endomorphism* $\mathcal{V}$ of the bundle J^1E:

$$\Theta_{\mathcal{L}} := i(\mathcal{V})\mathcal{L} + \mathcal{L} \equiv \theta_{\mathcal{L}} + \mathcal{L} \in \Omega^m(J^1E) \quad ; \quad \Omega_{\mathcal{L}} := -\mathrm{d}\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^1E)$$

In a natural chart in J^1E, in which $\bar{\pi}^{1^*}\omega = \mathrm{d}^m x$, we have

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial \pounds}{\partial v^A_\mu}\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \left(\frac{\partial \pounds}{\partial v^A_\mu}v^A_\mu - \pounds\right)\mathrm{d}^m x \\
\Omega_{\mathcal{L}} &= -\frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu}\mathrm{d}v^B_\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu}\mathrm{d}y^B \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \\
&\quad \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu}v^A_\mu \mathrm{d}v^B_\nu \wedge \mathrm{d}^m x + \left(\frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu}v^A_\mu - \frac{\partial \pounds}{\partial y^B} + \frac{\partial^2 \pounds}{\partial x^\mu \partial v^B_\mu}\right)\mathrm{d}y^B \wedge \mathrm{d}^m x
\end{aligned}$$

(For a more detailed description of all these concepts see, for instance, [1], [5], [7], [10], [28]).

For constructing the Hamiltonian formalism associated with a Lagrangian system in Field Theory, the *Legendre maps* are introduced. Then, depending on the choice of the multimomentum bundle, we can define different types of these maps, as follows:

Definition 7 *Let $((E,M;\pi),\mathcal{L})$ be a Lagrangian system, and $\bar{y} \in J^1E$, with $\bar{y} \overset{\pi^1}{\mapsto} y \overset{\pi}{\mapsto} x$.*

1. *Let $\mathcal{D} \subset \mathrm{T}J^1E$ be the subbundle of total derivatives in J^1E (which in a system of natural coordinates in J^1E, is generated by $\left\{\dfrac{\partial}{\partial x^\mu} + v^A_\mu \dfrac{\partial}{\partial y^A}\right\}$). We have that $\pi^{1*}\mathrm{T}E = \pi^{1*}\mathrm{V}(\pi) \oplus \mathcal{D}$ with $\mathrm{T}_yE|_{\bar{y}} = \mathrm{V}_y\pi|_{\bar{y}} \oplus \mathcal{D}_{\bar{y}}$ (see [28] for details). Hence there is a natural projection $\sigma\colon \pi^{1*}\mathrm{T}E \to \pi^{1*}\mathrm{V}(\pi)$, and we can draw the diagram*

$$\begin{tikzcd}
\mathrm{T}_{\bar{y}}J^1_yE = \mathrm{V}_{\bar{y}}\pi^1 \simeq (\mathrm{T}^*_xM \otimes \mathrm{V}_y\pi)_{\bar{y}} \arrow[r, "D_{\bar{y}}\mathcal{L}_y"] & (\Lambda^m\mathrm{T}^*_xM)_{\bar{y}} \\
(\mathrm{T}^*_xM \otimes \mathrm{T}_yE)_{\bar{y}} \arrow[u, "\mathrm{Id}\otimes\sigma_{\bar{y}}"'] \arrow[ur] &
\end{tikzcd}$$

   Then, the generalized Legendre map *is the* C^∞*-map*

$$\begin{array}{rccc}
\widehat{\mathrm{F}\mathcal{L}} : & J^1E & \to & J^1E^* \\
& \bar{y} & \mapsto & D_{\bar{y}}\mathcal{L}_y \circ (\mathrm{Id}\otimes\sigma)_{\bar{y}}
\end{array}$$

2. *The* reduced Legendre map *is the* C^∞*-map*

$$\begin{array}{rccc}
\mathrm{F}\mathcal{L} : & J^1E & \to & \Pi \\
& \bar{y} & \mapsto & \tilde{\mathrm{T}}_{\bar{y}}\mathcal{L}_y
\end{array}$$

where the map $\tilde{\mathrm{T}}_{\bar{y}}\mathcal{L}_y$ *is defined from the following diagram (where the vertical arrows are canonical isomorphisms given by the directional derivatives)*

$$\begin{array}{ccc}
\mathrm{T}_{\bar{y}}J^1_yE & \xrightarrow[\mathrm{T}_{\bar{y}}\mathcal{L}_y]{} & \mathrm{T}_{\mathcal{L}_y(\bar{y})}\Lambda^m\mathrm{T}^*_xM \\
\simeq\updownarrow & & \updownarrow\simeq \\
\mathrm{T}^*_xM\otimes\mathrm{V}_y\pi & \xrightarrow{\tilde{\mathrm{T}}_{\bar{y}}\mathcal{L}_y} & \Lambda^m\mathrm{T}^*_xM
\end{array}$$

($\mathrm{F}\mathcal{L}$ *is just the vertical derivative of* $\mathcal{L}$ *[13]).*

3. *The* (first) extended Legendre map *is the* C^∞*-map* $\widehat{\mathcal{FL}}\colon J^1E\to\mathcal{M}\pi$ *given by*

$$\widehat{\mathcal{FL}}:=\iota_0\circ\widehat{\mathrm{F}\mathcal{L}}$$

The (second) extended Legendre map *is the* C^∞*-map* $\widetilde{\mathcal{FL}}\colon J^1E\to\mathcal{M}\pi$ *given by*

$$\widetilde{\mathcal{FL}}=\widehat{\mathcal{FL}}+\pi^*\mathcal{L}$$

4. *The* restricted Legendre map *is the* C^∞*-map* $\mathcal{FL}\colon J^1E\to\Pi$ *given by*

$$\mathcal{FL}:=\mu\circ\widehat{\mathcal{FL}}=\mu\circ\widetilde{\mathcal{FL}}$$

If $\bar{y}\in J^1E$, with $\pi^1(\bar{y})=y$, using the natural coordinates in the different multimomentum bundles, we have:

$$\begin{aligned}
\mathcal{FL}(\bar{y}) &= \frac{\partial\pounds}{\partial v^A_\mu}(\bar{y})\mathrm{d}y^A\wedge\mathrm{d}^{m-1}x_\mu\Big|_y \\
\widetilde{\mathcal{FL}}(\bar{y}) &= \frac{\partial\pounds}{\partial v^A_\mu}(\bar{y})\mathrm{d}y^A\wedge\mathrm{d}^{m-1}x_\mu\Big|_y+(\pounds-v^A_\nu)(\bar{y})\frac{\partial\pounds}{\partial v^A_\mu}(\bar{y})\mathrm{d}^mx\Big|_y \\
\widehat{\mathcal{FL}}(\bar{y}) &= \frac{\partial\pounds}{\partial v^A_\mu}(\bar{y})\mathrm{d}y^A\wedge\mathrm{d}^{m-1}x_\mu\Big|_y-v^A_\nu(\bar{y})\frac{\partial\pounds}{\partial v^A_\mu}(\bar{y})\mathrm{d}^mx\Big|_y \\
\widehat{\mathrm{F}\mathcal{L}}(\bar{y}) &= \frac{\partial\pounds}{\partial v^A_\mu}(\bar{y})\left(\frac{\partial}{\partial x^\mu}\otimes(\mathrm{d}y^A-v^A_\nu(\bar{y})\mathrm{d}x^\nu)\otimes\mathrm{d}^mx\right)\Big|_y \\
\mathrm{F}\mathcal{L}(\bar{y}) &= \frac{\partial\pounds}{\partial v^A_\mu}(\bar{y})\left(\frac{\partial}{\partial x^\mu}\otimes\zeta^A\otimes\mathrm{d}^mx\right)\Big|_y
\end{aligned}$$

that is, the local expressions of the Legendre maps are the following:

$$\begin{array}{lllllll}
\mathcal{FL}^*x^\mu=x^\mu & , & \mathcal{FL}^*y^A=y^A & , & \mathcal{FL}^*p^\mu_A=\frac{\partial\pounds}{\partial v^A_\mu} & & \\
\widetilde{\mathcal{FL}}^*x^\mu=x^\mu & , & \widetilde{\mathcal{FL}}^*y^A=y^A & , & \widetilde{\mathcal{FL}}^*p^\mu_A=\frac{\partial\pounds}{\partial v^A_\mu} & , & \widetilde{\mathcal{FL}}^*p=\pounds-v^A_\mu\frac{\partial\pounds}{\partial v^A_\mu} \\
\widehat{\mathcal{FL}}^*x^\mu=x^\mu & , & \widehat{\mathcal{FL}}^*y^A=y^A & , & \widehat{\mathcal{FL}}^*p^\mu_A=\frac{\partial\pounds}{\partial v^A_\mu} & , & \widehat{\mathcal{FL}}^*p=-v^A_\mu\frac{\partial\pounds}{\partial v^A_\mu} \\
\widehat{\mathrm{F}\mathcal{L}}^*x^\mu=x^\mu & , & \widehat{\mathrm{F}\mathcal{L}}^*y^A=y^A & , & \widehat{\mathrm{F}\mathcal{L}}^*\mathrm{p}^\mu_A=\frac{\partial\pounds}{\partial v^A_\mu} & , & \widehat{\mathrm{F}\mathcal{L}}^*\mathrm{p}^\mu_\nu=-v^A_\nu\frac{\partial\pounds}{\partial v^A_\mu} \\
\mathrm{F}\mathcal{L}^*x^\mu=x^\mu & , & \mathrm{F}\mathcal{L}^*y^A=y^A & , & \mathrm{F}\mathcal{L}^*\mathrm{p}^\mu_A=\frac{\partial\pounds}{\partial v^A_\mu} & &
\end{array}$$

Remarks:

- Taking into account all the above results, it is immediate to prove that $\mathrm{F}\mathcal{L}=\delta\circ\widehat{\mathrm{F}\mathcal{L}}$ (see diagrams (2) and (4)), and $\mathrm{F}\mathcal{L}=\Psi\circ\mathcal{FL}$.

- It is interesting to point out that, as $\Theta_{\mathcal{L}}$ and $\theta_{\mathcal{L}}$ can be thought as m-forms on E along the projection $\pi^1\colon J^1E \to E$, the extended Legendre maps can be defined as

$$\begin{aligned} (\widehat{\mathcal{FL}}(\bar{y}))(Z_1,\ldots,Z_m) &= (\theta_{\mathcal{L}})_{\bar{y}}(\bar{Z}_1,\ldots,\bar{Z}_m) \\ (\widetilde{\mathcal{FL}}(\bar{y}))(Z_1,\ldots,Z_m) &= (\Theta_{\mathcal{L}})_{\bar{y}}(\bar{Z}_1,\ldots,\bar{Z}_m) \end{aligned}$$

  where $Z_1,\ldots,Z_m \in \mathrm{T}_{\pi^1(\bar{y})}E$, and $\bar{Z}_1,\ldots,\bar{Z}_m \in \mathrm{T}_{\bar{y}}J^1E$ are such that $\mathrm{T}_{\bar{y}}\pi^1\bar{Z}_\mu = Z_\mu$.

  In addition, the (second) extended Legendre map can also be defined as the "first order vertical Taylor approximation to $\pounds$" [3], [12].

Finally, we have the following relations between the Legendre maps and the Poincaré-Cartan $(m+1)$-form in J^1E (which can be easily proved using natural systems of coordinates and the expressions of the Legendre maps):

Proposition 4 *Let $((E,M;\pi),\mathcal{L})$ be a Lagrangian system. Then:*

$$\begin{aligned} \widetilde{\mathcal{FL}}^*\Theta = \Theta_{\mathcal{L}} \quad &; \quad \widetilde{\mathcal{FL}}^*\Omega = \Omega_{\mathcal{L}} \\ \widehat{\mathcal{FL}}^*\Theta = \Theta_{\mathcal{L}} - \mathcal{L} = \theta_{\mathcal{L}} \quad &; \quad \widehat{\mathcal{FL}}^*\Omega = \Omega_{\mathcal{L}} - \mathrm{d}\mathcal{L} = -\mathrm{d}\theta_{\mathcal{L}} \\ \widehat{\mathrm{F}\mathcal{L}}^*\hat{\Theta} = \Theta_{\mathcal{L}} - \mathcal{L} = \theta_{\mathcal{L}} \quad &; \quad \widehat{\mathrm{F}\mathcal{L}}^*\hat{\Omega} = \Omega_{\mathcal{L}} - \mathrm{d}\mathcal{L} = -\mathrm{d}\theta_{\mathcal{L}} \end{aligned}$$

Remark:

- Observe that the Hamiltonian formalism is essentially the dual formalism of the Lagrangian model, by means of the Lagrangian density. Then, as J^1E is an affine bundle, its affine dual can be identified with $\mathrm{Aff}\,(J^1E,\pi^*\Lambda^m\mathrm{T}^*M) \simeq \mathcal{M}\pi$, whose dimension is greater than $\dim J^1E$, and hence $\mathrm{Aff}\,(J^1E,\pi^*\Lambda^m\mathrm{T}^*M)/\Lambda^m_0\mathrm{T}^*E \simeq J^1\pi^*$ is more suitable as a dual bundle (from the dimensional point of view). Then, the canonical forms Θ and Ω in $\mathcal{M}\pi$, and $\hat{\Theta}$ and $\hat{\Omega}$ in J^1E^*, can be pulled-back to the restricted and reduced multimomentum bundles $J^1\pi^*$ and Π, using sections of the projections $\mu\colon\mathcal{M}\pi \to J^1\pi^*$ and $\delta\colon J^1E^* \to \Pi$, respectively [3], [6]. In this way, the reduced and restricted multimomentum bundles are endowed with (non-canonical) geometrical structures (*Hamilton-Cartan forms*) needed for stating the Hamiltonian formalism.

  In addition, connections in the bundle $\pi\colon E \to M$ induce linear sections of μ (and δ) [3], [6], [9], [25], and it can be proved that there is a bijective correspondence between the set of connections in the bundle $\pi\colon E \to M$, and the set of linear sections of the projection μ.

  Hence, all these results, together with Theorem 1, allows us to relate two of the most usual Hamiltonian formalisms of Field Theories [6].

5 Regular and singular systems

Following the well-known terminology of mechanics, we define:

Definition 8 *Let $((E,M;\pi),\mathcal{L})$ be a Lagrangian system.*

1. *$((E,M;\pi),\mathcal{L})$ is said to be a* regular *or* non-degenerate *Lagrangian system if $\mathcal{FL}$, and hence, $\mathrm{F}\mathcal{L}$ are local diffeomorphisms.*

   As a particular case, $((E,M;\pi),\mathcal{L})$ is said to be a hyper-regular *Lagrangian system if $\mathcal{FL}$, and hence $\mathrm{F}\mathcal{L}$, are global diffeomorphisms.*

2. *Elsewhere* $((E,M;\pi),\mathcal{L})$ *is said to be a* singular *or* degenerate *Lagrangian system.*

Proposition 5 *Let* $((E,M;\pi),\mathcal{L})$ *a hyper-regular Lagrangian system. Then:*

1. $\widehat{\mathrm{F}\mathcal{L}}(J^1E)$ *is a* m^2*-codimensional imbedded submanifold of* J^1E^* *which is transverse to the projection* δ.
2. $\widehat{\mathcal{FL}}(J^1E)$ *and* $\widetilde{\mathcal{FL}}(J^1E)$ *are 1-codimensional imbedded submanifolds of* $\mathcal{M}\pi$ *which are transverse to the projection* μ.
3. *The manifolds* $J^1\pi^*$, $\widehat{\mathcal{FL}}(J^1E)$, $\widetilde{\mathcal{FL}}(J^1E)$, $\widehat{\mathrm{F}\mathcal{L}}(J^1E)$ *and* Π *are diffeomorphic.*
   Hence, $\widehat{\mathrm{F}\mathcal{L}}$, $\widehat{\mathcal{FL}}$ *and* $\widetilde{\mathcal{FL}}$ *are diffeomorphisms on their images; and the maps* μ, *restricted to* $\widehat{\mathcal{FL}}(J^1E)$ *or to* $\widetilde{\mathcal{FL}}(J^1E)$, *and* ι_0 *and* δ, *restricted to* $\widehat{\mathrm{F}\mathcal{L}}(J^1E)$, *are also diffeomorphisms.*

(*Proof*)

1. If $\mathcal{L}$ is hyper-regular then $\mathrm{F}\mathcal{L}$ is a diffeomorphism and hence, as $\mathrm{F}\mathcal{L} = \widehat{\mathrm{F}\mathcal{L}} \circ \delta$, we obtain that $\widehat{\mathrm{F}\mathcal{L}}$ is injective and $\widehat{\mathrm{F}\mathcal{L}}(J^1E)$ is transverse to the fibers of δ.
2. The proof of this statement is like for the one above (see also [21]).
3. It is a direct consequence of the above items.

■

In this way we have the following (commutative) diagram

$$
\begin{tikzcd}
 & J^1\pi^* \arrow[dddd, "\Psi"] \\
 & \mathcal{M}\pi \arrow[u, "\mu"'] \\
J^1E \arrow[uur, "\mathcal{FL}"] \arrow[ur, "\widetilde{\mathcal{FL}}"] \arrow[r, "\widehat{\mathcal{FL}}"] \arrow[dr, "\widehat{\mathrm{F}\mathcal{L}}"'] \arrow[ddr, "\mathrm{F}\mathcal{L}"'] & \mathcal{M}\pi \arrow[uu, "\mu"'] \\
 & J^1E^* \arrow[u, "\iota_0"'] \arrow[d, "\delta"'] \\
 & \Pi
\end{tikzcd}
\tag{2}
$$

Observe that there exists a map $\mu' \colon \widehat{\mathcal{FL}}(J^1E) \subset \mathcal{M}\pi \to \widetilde{\mathcal{FL}}(J^1E) \subset \mathcal{M}\pi$, which is a diffeomorphism defined by the relation $\widetilde{\mathcal{FL}} = \mu' \circ \widehat{\mathcal{FL}}$, and $\mu \circ \mu' = \mu$ on $\widehat{\mathcal{FL}}(J^1E)$.

For dealing with singular Lagrangians we must assume minimal "regularity" conditions. Hence we introduce the following terminology:

Definition 9 *A singular Lagrangian system* $((E,M;\pi),\mathcal{L})$ *is said to be* almost-regular *if:*

1. $\mathcal{P} := \mathcal{FL}(J^1E)$ *and* $P := \mathrm{F}\mathcal{L}(J^1E)$ *are closed submanifolds of* $J^1\pi^*$ *and* Π, *respectively.*
   (We will denote by $\jmath_0 \colon \mathcal{P} \hookrightarrow J^1\pi^*$ *and* $\mathrm{j}_0 \colon P \hookrightarrow \Pi$ *the corresponding imbeddings).*

2. *$\mathcal{FL}$, and hence $\mathrm{F}\mathcal{L}$, are submersions onto their images.*
3. *For every $\bar{y} \in J^1E$, the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y}))$ and hence $\mathrm{F}\mathcal{L}^{-1}(\mathrm{F}\mathcal{L}(\bar{y}))$ are connected submanifolds of J^1E.*

(This definition is equivalent to that in reference [21], but slightly different from that in references [9] and [25]).

Let $((E, M; \pi), \mathcal{L})$ be an almost-regular Lagrangian system. Denote

$$\hat{\mathcal{P}} := \widehat{\mathcal{FL}}(J^1E) \quad , \quad \tilde{\mathcal{P}} := \widetilde{\mathcal{FL}}(J^1E) \quad , \quad \hat{P} := \widehat{\mathrm{F}\mathcal{L}}(J^1E)$$

Let $\hat{\jmath}_0\colon \hat{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$, $\tilde{\jmath}_0\colon \tilde{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$, $\hat{\mathrm{j}}_0\colon \hat{P} \hookrightarrow J^1E^*$ be the canonical inclusions, and

$$\hat{\mu}\colon \hat{\mathcal{P}} \to \mathcal{P} \quad , \quad \tilde{\mu}\colon \tilde{\mathcal{P}} \to \mathcal{P} \quad , \quad \hat{\iota}_0\colon \hat{P} \to \hat{\mathcal{P}} \quad , \quad \hat{\delta}\colon \hat{P} \to P \quad , \quad \Psi_0\colon \mathcal{P} \to P$$

the restrictions of the maps μ, ι_0, δ and the diffeomorphism Ψ, respectively. Finally, define the restriction mappings

$$\mathcal{FL}_0\colon J^1E \to \mathcal{P} \ , \ \widetilde{\mathcal{FL}}_0\colon J^1E \to \tilde{\mathcal{P}} \ , \ \widehat{\mathcal{FL}}_0\colon J^1E \to \hat{\mathcal{P}} \ , \ \widehat{\mathrm{F}\mathcal{L}}_0\colon J^1E \to \hat{P} \ , \ \mathrm{F}\mathcal{L}_0\colon J^1E \to P$$

Proposition 6 *Let $((E, M; \pi), \mathcal{L})$ be an almost-regular Lagrangian system. Then:*

1. *The maps Ψ_0 and $\tilde{\mu}$ are diffeomorphisms.*
2. *For every $\bar{y} \in J^1E$,*
$$\widetilde{\mathcal{FL}_0}^{-1}(\widetilde{\mathcal{FL}_0}(\bar{y})) = \mathcal{FL}_0^{-1}(\mathcal{FL}_0(\bar{y})) = \mathrm{F}\mathcal{L}_0^{-1}(\mathrm{F}\mathcal{L}_0(\bar{y})) \tag{3}$$
3. *$\tilde{\mathcal{P}}$ and $\hat{\mathcal{P}}$ are submanifolds of $\mathcal{M}\pi$, $\hat{P}$ is a submanifold of J^1E^*, and $\tilde{\jmath}_0\colon \tilde{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$, $\hat{\jmath}_0\colon \hat{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$, $\hat{\mathrm{j}}_0\colon \hat{P} \hookrightarrow J^1E^*$ are imbeddings.*
4. *The restriction mappings $\widetilde{\mathcal{FL}}_0$, $\widehat{\mathcal{FL}}_0$ and $\widehat{\mathrm{F}\mathcal{L}}_0$ are submersions with connected fibers.*

(*Proof*) Ψ_0 is a diffeomorphism as is Ψ.

The second equality of (3) is a consequence of the relation $\mathrm{F}\mathcal{L}_0 = \Psi_0 \circ \mathcal{FL}_0$.

For the proof of the first equality of (3), and of the assertions concerning $\tilde{\mu}$, $\tilde{\mathcal{P}}$ and $\widetilde{\mathcal{FL}}_0$, see [21] and [22]. Then, the proofs of the other assertions are similar. ■

Thus we have the (commutative) diagram

$$\begin{tikzcd}
 & \mathcal{P} \arrow[dddd, bend left, "\Psi_0"] \arrow[rr, "\jmath_0"] & & J^1\pi^* \arrow[dddd, bend left, "\Psi"] \\
 & \tilde{\mathcal{P}} \arrow[u, "\tilde{\mu}"] \arrow[rr, "\tilde{\jmath}_0"] & & \mathcal{M}\pi \arrow[u, "\mu"] \\
J^1E \arrow[uur, "\mathcal{FL}_0"] \arrow[ur, "\widetilde{\mathcal{FL}}_0"] \arrow[r, "\widehat{\mathcal{FL}}_0"] \arrow[dr, "\widehat{\mathrm{F}\mathcal{L}}_0"] \arrow[ddr, "\mathrm{F}\mathcal{L}_0"] & \hat{\mathcal{P}} \arrow[u, "\hat{\mu}'"] \arrow[uu, bend left, "\hat{\mu}"] \arrow[rr, "\hat{\jmath}_0"] & & \mathcal{M}\pi \arrow[uu, bend left, "\mu"] \\
 & \hat{P} \arrow[u, "\hat{\iota}_0"] \arrow[d, "\hat{\delta}"] \arrow[rr, "\hat{\mathrm{j}}_0"] & & J^1E^* \arrow[u, "\iota_0"] \arrow[d, "\delta"] \\
 & P \arrow[rr, "\mathrm{j}_0"] & & \Pi
\end{tikzcd} \tag{4}$$

where $\hat{\mu}': \hat{\mathcal{P}} \to \tilde{\mathcal{P}}$ is defined by the relation $\hat{\mu}' := \tilde{\mu}^{-1} \circ \hat{\mu}$.

Remarks:

- The fact that $\tilde{\mu}$ is a diffeomorphism is particularly relevant, since it allows us to construct a Hamiltonian formalism for an almost-regular Lagrangian system [22].
- It is interesting to point out that the map $\hat{\mu}$ (which is related with the Legendre map $\widehat{\mathcal{FL}_0}$) is not a diffeomorphism in general, since $\mathrm{rank}\,\widehat{\mathcal{FL}_0} \geq \mathrm{rank}\,\widetilde{\mathcal{FL}_0} = \mathrm{rank}\,\mathcal{FL}_0$, as is evident from the analysis of the corresponding Jacobian matrices.

The matrix of the tangent maps $\mathcal{FL}_*$ and $\mathrm{F}\mathcal{L}_*$ in a natural coordinate system is

$$\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ \frac{\partial^2 \pounds}{\partial x^\nu \partial v^A_\mu} & \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu} & \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu} \end{pmatrix}$$

where the sub-matrix $\left(\frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu} \right)$ is the *partial Hessian matrix* of $\mathcal{L}$. Obviously, the regularity of $\mathcal{L}$ is equivalent to demanding that the partial Hessian matrix $\left(\frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu} \right)$ is regular everywhere in J^1E. This fact establishes the relation to the concept of regularity given in an equivalent way by saying that a Lagrangian system $((E, M; \pi), \mathcal{L})$ is *regular* if $\Omega_{\mathcal{L}}$ is 1-nondegenerate (elsewhere it is said to be *singular* or *non-regular*).

Conclusions

- We have reviewed the definitions of four different multimomentum bundles for the Hamiltonian formalism of first-order Classical Field Theories (multisymplectic models). The so-called *generalized* and *reduced* multimomentum bundles are related straightforward from their definition, and the same thing happens with the *generalized* and *restricted* multimomentum bundles. The first goal of this work has been to relate both couples, proving that the *reduced* and *restricted* multimomentum bundles are, in fact, canonically diffeomorphic. In natural local coordinates, this diffeomorphism is just the identity.
- The canonical forms which the generalized and the extended multimomentum bundles are endowed with, have been defined and characterized in several equivalent ways.
- Given a *Lagrangian system* in Field Theory, we have introduced the corresponding *Legendre maps* relating these multimomentum bundles to the first-order jet bundle associated with this system. Some of them, the *generalized* and *reduced* Legendre maps, are defined in a natural way as *fiber derivatives* of the Lagrangian density, being the other ones obtained from those. The relation among all these maps has been clarified.
- *Regular* and *almost-regular* Lagrangian systems are defined and studied, attending to the geometric features of the Legendre maps. In this way, the standard definitions existing in the usual literature are extended and completed.

A Appendix

Lemma 1 *Let $\pi\colon F \to N$ be a differentiable bundle, with $\dim N = n$ and $\dim F = n + r$, and $p \in F$ with $q = \pi(p)$. Let $v_1, \ldots, v_h \in \mathrm{T}_pF$ $(h \leq n)$, such that: $v_1, \ldots, v_h$ are linearly independent, and $\langle v_1, \ldots, v_h\rangle \cap \mathrm{V}_p(\pi) = \{0\}$. Then:*

1. *There exist $X_1, \ldots, X_h \in \mathfrak{X}(W)$, for a neighborhood $W \subset F$ of p, such that:*
   - *(a) $X_1, \ldots, X_h$ are linearly independent, at every point of W.*
   - *(b) $X_1, \ldots, X_h$ generate an involutive distribution in W.*
   - *(c) $X_i(p) = v_i$, $(i = 1, \ldots, h)$.*
   - *(d) $\langle X_1(x), \ldots, X_h(x)\rangle \cap \mathrm{V}_x(\pi) = \{0\}$, for every $x \in W$.*
2. *There exists a local section γ of π, defined in a neighborhood of $q \in N$, and $u_1, \ldots, u_h \in \mathrm{T}_qN$ such that: $\gamma(q) = p$, and $\mathrm{T}_p\gamma(u_i) = v_i$, $(i = 1, \ldots, h)$.*

(*Proof*) Let $v_{h+1}, \ldots, v_n \in \mathrm{T}_pF$, such that $\mathrm{T}_p = \mathrm{V}_p(\pi) \oplus \langle v_1, \ldots, v_n\rangle$. Let (W, φ) be a local chart of F at p, adapted to π. We have $\varphi\colon W \to U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^r$. Let $e_1, \ldots, e_n$ and $e_{n+1}, \ldots, e_{n+r}$ be local basis of $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively. Then

$$\begin{aligned} \langle \mathrm{T}_p\varphi(v_1), \ldots, \mathrm{T}_p\varphi(v_n)\rangle &= \langle e_1, \ldots, e_n\rangle \\ \mathrm{T}_x\varphi(\mathrm{V}_x(\pi)) &= \langle e_{n+1}, \ldots, e_{n+r}\rangle \quad , \quad \forall x \in W \end{aligned} \tag{5}$$

Let $Z_1, \ldots, Z_n \in \mathfrak{X}(U_1 \times U_2)$ be the constant extensions of $\mathrm{T}_p\varphi(v_1), \ldots, \mathrm{T}_p\varphi(v_n)$ to $U_1 \times U_2$. We have that $[Z_i, Z_j] = 0$, $\forall i, j$. Finally, let $X_1, \ldots, X_n \in \mathfrak{X}(W)$, with $X_i = \varphi^* Z_i$, therefore:

1. Taking $X_1, \ldots, X_h$, $(h \leq n)$, conditions (1.a), (1.b) and (1.c) hold trivially (by construction), and condition (1.d) holds as a consequence of (5).
2. Observe that $X_1, \ldots, X_n \in \mathfrak{X}(W)$ generate the horizontal subspace of a connection defined in the bundle $\pi\colon W \to \pi(W)$, which is integrable because the distribution is involutive. Then, let γ be the integral section of this connection at p. Therefore $\gamma(q) = p$, and the subspace tangent to the image of γ at p is generated by $v_1, \ldots, v_n$. Hence, there exist $u_1, \ldots, u_n$ such that $\mathrm{T}_q\gamma(u_i) = v_i$.

■

Now, let A be an affine space modeled on a vector space S, and T another vector space, both over the same field K. Let $\mathrm{Aff}(A, T)$ be the set of affine maps from A to T; that is,

maps $\varphi\colon A \to T$ such that there exists a linear map $\hat{\varphi}\colon S \to T$ verifying that $\varphi(a) - \varphi(b) = \hat{\varphi}(a - b)$; for $a, b \in A$. Then:

Lemma 2 *1. There is a natural isomorphism between $\mathrm{Aff}(A, T)/T$ and $S^* \otimes T$ (and then $\dim \mathrm{Aff}(A, T) = \dim (S^* \otimes T) + \dim T = \dim T(\dim S + 1)$).*

2. There is a canonical isomorphism between $\mathrm{Aff}(A, T)$ and $\mathrm{Aff}(A, K) \otimes T$.

(*Proof*) $\mathrm{Aff}(A, T)$ is a vector space over K with the natural operations. The map $\wedge\colon \mathrm{Aff}(A, T) \to S^* \otimes T$, which assigns $\hat{\varphi}$ to every φ, is linear and we have the exact sequence

$$0 \longrightarrow T \stackrel{j}{\longrightarrow} \mathrm{Aff}(A, T) \stackrel{\wedge}{\longrightarrow} S^* \otimes T \longrightarrow 0$$

where, if $t \in T$, then $j(t)\colon A \to T$ is the constant map $(j(t))(a) = t$, for every $a \in A$. Therefore we have a natural isomorphism $\mathrm{Aff}(A,T)/T \simeq S^* \otimes T$ and then

$$\dim\,\mathrm{Aff}(A,T) = \dim\,(S^* \otimes T) + \dim\,T = \dim\,T(\dim\,S + 1)$$

Moreover, for $y_0 \in A$, there exists an splitting $\mathrm{Aff}\,(A,T) \simeq T \oplus (S^* \otimes T)$ given by the following retract of the above exact sequence

$$\begin{array}{cccccc} j_{y_0} & : & S^* \otimes T & \to & \mathrm{Aff}\,(A,T) & \\ & & \varphi & \mapsto & \varphi_0\colon & y \mapsto \varphi(y - y_0) \end{array}$$

On the other hand, we have the bilinear map

$$\begin{array}{ccccc} \mathrm{Aff}(A,K) \times T & \longrightarrow & \mathrm{Aff}(A,T) & \\ (\alpha, t) & \mapsto & t\alpha : & a \mapsto \alpha(a)t \end{array}$$

and hence we can define the following morphism

$$\begin{array}{ccc} \mathrm{Aff}(A,K) \otimes T & \longrightarrow & \mathrm{Aff}(A,T) \\ \alpha^i \otimes u_i & \mapsto & u_i\alpha^i \end{array}$$

which is injective because we can assume that the vectors u_i are linearly independent, and both spaces have the same dimension. Therefore $\mathrm{Aff}(A,T)$ and $\mathrm{Aff}(A,K) \otimes T$ are canonically isomorphic.

■

If $(s_1, \ldots, s_m)$ is a basis of S, $(\sigma^1, \ldots, \sigma^m)$ is its dual basis and $(t_1, \ldots, t_m)$ is a basis of T, then taking an affine reference in A, $(t_i, t_i \otimes \sigma^j)$ is a basis of $\mathrm{Aff}(A,T)$, as vector space.

Now, let G, H be finite dimensional vector spaces over K, and F a subspace of G. Consider the exact sequence

$$0 \longrightarrow F \xrightarrow{\tau} G \xrightarrow{\pi} H \longrightarrow 0 \tag{6}$$

The set

$$\Sigma \equiv \{\sigma\colon H \to G \;;\; \sigma \text{ linear }, \pi \circ \sigma = \mathrm{Id}_H\}$$

is an affine space modeled on $L(H,F) = H^* \otimes F$. In fact, if $\sigma \in \Sigma$ and $\lambda \in H^* \otimes F$, then $\sigma + \lambda \in \Sigma$, since $\pi \circ \lambda = 0$. Furthermore, if $\sigma, \mu \in \Sigma$, then $\pi \circ (\sigma - \mu) = 0$, and hence $\sigma - \mu \in H^* \otimes F$.

If $\dim\,H = m$, taking the space $\mathrm{Aff}(\Sigma, \Lambda^m H^*)$, and

according to the second item of the above Lemma, we have that $\mathrm{Aff}(\Sigma, \Lambda^m H^*) \simeq \mathrm{Aff}(\Sigma, K) \otimes \Lambda^m H^*$, and then $\dim\,\mathrm{Aff}(\Sigma, \Lambda^m H^*) = \dim\,\mathrm{Aff}(\Sigma, K) = \dim\,(H^* \otimes F) + 1$. Now, consider the subspace

$$\Lambda^m_1 G^* \equiv \{\alpha \in \Lambda^m G^* \;:\; i(u)\,i(v)\alpha = 0 \;,\; u, v \in F\} \subset \Lambda^m G^*$$

Lemma 3 *The spaces* $\mathrm{Aff}(\Sigma, \Lambda^m H^*)$ *and* $\Lambda^m_1 G^*$ *are canonically isomorphic.*

(*Proof*) If $(f^1, \ldots, f^r)$ is a basis of F and $g^1, \ldots, g^m \in G$ such that $(f^1, \ldots, f^r, g^1, \ldots, g^m)$ is a basis of G, then $\Lambda^m_1 G^*$ is generated by $(g^1 \wedge \ldots \wedge g^m, f^j \wedge g^{i_1} \wedge \ldots \wedge g^{i_{m-1}})$ (with $1 \leq i_1 < \ldots < i_{m-1} < m$), therefore $\dim\,\Lambda^m_1 G^* = 1 + \dim\,F\,\dim\,H$; that is, $\dim\,\Lambda^m_1 G^* = \dim\,\mathrm{Aff}(\Sigma, \Lambda^m H^*)$. Next, define the linear map

$$\begin{array}{cccccc} \Upsilon & : & \Lambda^m_1 G^* & \longrightarrow & \mathrm{Aff}(\Sigma, \Lambda^m H^*) & \\ & & \eta & \mapsto & \Upsilon(\eta) & : \;\sigma \mapsto \sigma^*\eta \end{array}$$

which we want to prove is an isomorphism, for which it suffices to prove that it is injective. Thus, suppose that $\Upsilon(\eta) = 0$ (that is $\sigma^*\eta = 0$, for every $\sigma \in \Sigma$), then we must prove that $\eta = 0$. Let $g^1, \ldots, g^m \in G$ be linearly independent (so $\dim\,L(g^1, \ldots, g^m) = m$); we are going to calculate $\eta(g^1, \ldots, g^m)$.

1. If $L(g^1,\ldots,g^m)\cap\tau(F)=\{0\}$:

   Then $G=L(g^1,\ldots,g^m)\oplus\tau(F)$ and therefore $\pi\colon L(g^1,\ldots,g^m)\to H$ is an isomorphism. Hence, there exist $h^1,\ldots,h^m\in H$ and $\sigma\colon H\to G$, with $\pi\circ\sigma=\mathrm{Id}_H$, such that $\sigma(h^i)=g^i$; therefore
   $$\eta(g^1,\ldots,g^m)=\eta(\sigma(h^1),\ldots,\sigma(h^m))=(\sigma^*\eta)(h^1,\ldots,h^m)=0$$

2. If $L(g^1,\ldots,g^m)\cap\tau(F)\neq\{0\}$:

   (a) If $\dim\left(L(g^1,\ldots,g^m)\cap\tau(F)\right)=1$:
   In this case there is $u\in L(g^1,\ldots,g^m)\cap\tau(F)$, with $u\neq 0$, such that $L(g^1,\ldots,g^m)=L(u,g^2,\ldots,g^m)$ (up to an ordering change), and then there is $k\in K$ such that
   $$\eta(g^1,\ldots,g^m)=k\eta(u,g^2,\ldots,g^m)$$
   therefore $L(g^2,\ldots,g^m)\cap\tau(F)=\{0\}$. Let $(u,u^1,\ldots,u^r)$ be a basis of F, and $\bar g^1$ such that $(u,u^1,\ldots,u^r,\bar g^1,g^2,\ldots,g^m)$ is a basis of G. Then $L(\bar g^1,g^2,\ldots,g^m)\cap\tau(F)=\{0\}$, and hence, as a consequence of the above item, we conclude that $\eta(\bar g^1,g^2,\ldots,g^m)=0$. Furthermore, $L(\bar g^1+u,g^2,\ldots,g^m)\cap\tau(F)=\{0\}$ because, if this is not true, then $L(\bar g^1,g^2,\ldots,g^m)\cap\tau(F)\neq\{0\}$, and hence $\eta(\bar g^1+u,g^2,\ldots,g^m)=0$, by the above item. Thus
   $$\eta(g^1,\ldots,g^m)=k\eta(u,g^2,\ldots,g^m)=k(\eta(\bar g^1+u,g^2,\ldots,g^m)-\eta(\bar g^1,g^2,\ldots,g^m))=0$$

   (b) If $\dim\left(L(g^1,\ldots,g^m)\cap\tau(F)\right)=s>1$, with $s\leq r=\dim F$:
   Let $(f^1,\ldots,f^s,g^{s+1},\ldots,g^m)$ be a basis of $L(g^1,\ldots,g^m)$, with $f^1,\ldots,f^s\in F$. Then, as $\eta\in\Lambda^mG^*$,
   $$\eta(g^1,\ldots,g^m)=k\eta(f^1,\ldots,f^s,g^{s+1},\ldots,g^m)=0$$

Then $\eta=0$, so Υ is injective and is a canonical isomorphism between $\mathrm{Aff}(\Sigma,\Lambda^mH^*)$ and $\Lambda^m_1G^*$. ∎

Using the first item of Lemma 2 and identifying $A=\Sigma$, we conclude:

Lemma 4 *$\mathrm{Aff}(\Sigma,\Lambda^mH^*)/\Lambda^mH^*\simeq(H^*\otimes F)^*\otimes\Lambda^mH^*\simeq H\otimes F^*\otimes\Lambda^mH^*$.*

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